The Fourier Series
The Fourier Expansion

Gary Schurman, MBE, CFA

September, 2023

A Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines. The base equation for a Fourier series is...

$$f(x) = A_0 + A_1 \cos(x) + B_1 \sin(x) + A_2 \cos(2\,x) + B_2 \sin(2\,x) + A_3 \cos(3\,x) + B_3 \sin(3\,x) + ... \tag{1}$$

Note that a periodic function is a function that repeats its values after every interval.

We can rewrite Equation (1) above as the following infinite series...

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\,x) + \sum_{m=1}^{\infty} B_m \sin(m\,x) \tag{2}$$

Note that for Fourier series the values of m and n in Equation (2) above will be positive non-zero integers.

Constant Term

To derive the value of the constant term A_0 in Equation (2) above we will take the integral of both sides of that equation over the periodic interval $[0, 2\,\pi]$ (note that the sine and cosine functions have a period of $2\,\pi$)...

$$\int_0^{2\,\pi} f(x)\,\delta x = \int_0^{2\,\pi} A_0\,\delta x + \sum_{n=1}^{\infty} A_n \int_0^{2\,\pi} \cos(n\,x)\,\delta x + \sum_{m=1}^{\infty} B_m \int_0^{2\,\pi} \sin(m\,x)\,\delta x \tag{3}$$

Note the solutions to the following integrals... [1]

$$\int_0^{2\,\pi} \sin(n\,x)\,\delta x = 0 \;\; \text{...and...} \;\; \int_0^{2\,\pi} \cos(m\,x)\,\delta x = 0 \tag{4}$$

Using the integral solutions in Equation (4) above, we can rewrite Equation (3) above as...

$$\int_0^{2\,\pi} f(x)\,\delta x = \int_0^{2\,\pi} A_0\,\delta x + \sum_{n=1}^{\infty} A_n \times 0 + \sum_{m=1}^{\infty} B_m \times 0 = A_0 \int_0^{2\,\pi} \delta x \tag{5}$$

The solution to Equation (5) above is...

$$\int_0^{2\,\pi} f(x)\,\delta x = A_0 \Big(2\,\pi - 0\Big) = A_0\,2\,\pi \tag{6}$$

Note that we can rewrite Equation (6) above as...

$$A_0 = \frac{1}{2\,\pi} \int_0^{2\,\pi} f(x)\,\delta x = \text{Average value of } f(x) \text{ over interval } [0, 2\,\pi] \tag{7}$$

Cosine Terms

To derive the value of the cosine terms B_m in Equation (2) above we will multiply both sides of that equation by $\cos(m\,x)$...

$$f(x)\,\cos(m\,x) = A_0\,\cos(m\,x) + \sum_{n=1}^{\infty} A_n \cos(n\,x)\,\cos(mx) + \sum_{m=1}^{\infty} B_m \sin(m\,x)\,\cos(mx) \tag{8}$$

We will then take the integral of both sides of Equation (8) above...

$$\int_0^{2\pi} f(x)\,\cos(m\,x)\,\delta x = A_0 \int_0^{2\pi} \cos(m\,x)\,\delta x + \sum_{n=1}^{\infty} A_n \int_0^{2\pi} \cos(n\,x)\,\cos(mx)\,\delta x + \sum_{m=1}^{\infty} B_m \int_0^{2\pi} \sin(m\,x)\,\cos(m\,x)\,\delta x \tag{9}$$

Note the solutions to the following integrals... [1]

$$\int_0^{2\pi} \cos(m\,x)\,\delta x = 0 \text{ ...and... } \int_0^{2\pi} \sin(n\,x)\,\cos(m\,x)\,\delta x = 0 \tag{10}$$

Using the integral solutions in Equation (10) above, we can rewrite Equation (9) above as...

$$\begin{aligned} \int_0^{2\pi} f(x)\,\cos(m\,x)\,\delta x &= A_0 \times 0 + \sum_{n=1}^{\infty} A_n \int_0^{2\pi} \cos(n\,x)\,\cos(mx)\,\delta x + \sum_{m=1}^{\infty} B_m \times 0 \\ &= \sum_{n=1}^{\infty} A_n \int_0^{2\pi} \cos(n\,x)\,\cos(mx)\,\delta x \end{aligned} \tag{11}$$

Note that we can rewrite Equation (11) above as...

$$\begin{aligned} \int_0^{2\pi} f(x)\,\cos(m\,x)\,\delta x = &\sum_{n=1}^{\infty} A_n \int_0^{2\pi} \cos(n\,x)\,\cos(m\,x)\,\delta x \text{ ...when...} n \neq m \\ &+ A_n \int_0^{2\pi} \cos(n\,x)\,\cos(mx)\,\delta x \text{ ...when...} n = m \end{aligned} \tag{12}$$

Note the solutions to the following integrals... [1]

$$\int_0^{2\pi} \cos(n\,x)\,\cos(m\,x)\,\delta x = 0 \text{ ...when... } n \neq m \text{ ...and... } \int_0^{2\pi} \cos(n\,x)\,\cos(m\,x)\,\delta x = \pi \text{ ...when... } n = m \tag{13}$$

Using the integral solutions in Equation (13) above, we can rewrite Equation (12) above as...

$$\int_0^{2\pi} f(x)\,\cos(m\,x)\,\delta x = \sum_{n=1}^{\infty} A_n \times 0 + A_m\,\pi = A_m\,\pi \tag{14}$$

Note that we can rewrite Equation (14) above as...

$$A_m = \frac{1}{\pi} \int_0^{2\pi} f(x)\,\cos(m\,x)\,\delta x \tag{15}$$

Sine Terms

To derive the value of the sine terms A_n in Equation (2) above we will multiply both sides of that equation by $\sin(m\,x)$...

$$f(x)\,\sin(n\,x) = A_0\,\sin(n\,x) + \sum_{n=1}^{\infty} A_n \cos(n\,x)\,\sin(nx) + \sum_{m=1}^{\infty} B_m \sin(m\,x)\,\sin(n\,x) \tag{16}$$

We will then take the integral of both sides of Equation (16) above...

$$\int_0^{2\pi} f(x)\,\sin(n\,x)\,\delta x = A_0 \int_0^{2\pi} \sin(n\,x)\,\delta x + \sum_{n=1}^{\infty} A_n \int_0^{2\pi} \cos(n\,x)\,\sin(n\,x)\,\delta x + \sum_{m=1}^{\infty} B_m \int_0^{2\pi} \sin(m\,x)\,\sin(n\,x)\,\delta x \tag{17}$$

Note the solutions to the following integrals... [1]

$$\int_0^{2\pi} \sin(m\,x)\,\delta x = 0 \;\text{...and...}\; \int_0^{2\pi} \sin(n\,x)\,\cos(m\,x)\,\delta x = 0 \tag{18}$$

Using the integral solutions in Equation (18) above, we can rewrite Equation (17) above as...

$$\begin{aligned}\int_0^{2\pi} f(x)\,\sin(n\,x)\,\delta x &= A_0 \times 0 + \sum_{n=1}^{\infty} A_n \times 0 + \sum_{m=1}^{\infty} B_m \int_0^{2\pi} \sin(m\,x)\,\sin(n\,x)\,\delta x \\ &= \sum_{m=1}^{\infty} B_m \int_0^{2\pi} \sin(m\,x)\,\sin(n\,x)\,\delta x \end{aligned} \tag{19}$$

Note that we can rewrite Equation (19) above as...

$$\begin{aligned}\int_0^{2\pi} f(x)\,\sin(m\,x)\,\delta x = \sum_{m=1}^{\infty} B_m \int_0^{2\pi} \sin(m\,x)\,\sin(n\,x)\,\delta x \;\text{...when...}\, n \neq m \\ + B_m \int_0^{2\pi} \sin(n\,x)\,\sin(mx)\,\delta x \;\text{...when...}\, n = m \end{aligned} \tag{20}$$

Note the solutions to the following integrals... [1]

$$\int_0^{2\pi} \sin(n\,x)\,\sin(m\,x)\,\delta x = 0 \;\text{...when...}\; n \neq m \;\text{...and...}\; \int_0^{2\pi} \sin(n\,x)\,\sin(m\,x)\,\delta x = \pi \;\text{...when...}\; n = m \tag{21}$$

Using the integral solutions in Equation (21) above, we can rewrite Equation (20) above as...

$$\int_0^{2\pi} f(x)\,\sin(m\,x)\,\delta x = \sum_{m=1}^{\infty} B_m \times 0 + B_m\,\pi = B_m\,\pi \tag{22}$$

Note that we can rewrite Equation (22) above as...

$$B_m = \frac{1}{\pi} \int_0^{2\pi} f(x)\,\sin(m\,x)\,\delta x \tag{23}$$

References

[1] Gary Schurman, *The Fourier Series - Derivatives and Integrals Of Trigonometric Functions,* September, 2023.